Credit to Dangna Li for these solutions.

1

Let $U = \frac{X}{X+Y}$, $V = \frac{X+Y}{X+Y+Z}$, $W = X + Y + Z$, then the joint probability density function of U, V, W is:

$$f_{u,v,w}(u, v, w) = f_{x,y,z}(x(u, v, w), y(u, v, w), z(u, v, w)) \cdot |J|$$

where

$$|J| = \begin{bmatrix} VW & UW & UV \\ -WV & W(1-U) & V(1-U) \\ 0 & -W & (1-V) \end{bmatrix} = W^2 V$$

Therefore

$$\begin{aligned} f_{u,v,w}(u, v, w) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} e^{-uvw} e^{-wv(1-u)} e^{-w(1-v)} (uvw)^{\alpha-1} \left(wv(1-u)\right)^{\beta-1} \left(w(1-v)\right)^{\gamma-1} \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} u^{\alpha-1}(1-u)^{\beta-1} v^{\alpha+\beta-1}(1-v)^{\gamma-1} e^{-w} w^{\alpha+\beta+\gamma-1} \\ &= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} u^{\alpha-1}(1-u)^{\beta-1} \\ &\quad \cdot \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha+\beta)\Gamma(\gamma)} v^{\alpha+\beta-1}(1-v)^{\gamma-1} \\ &\quad \cdot \frac{1}{\Gamma(\alpha+\beta+\gamma)} e^{-w} w^{\alpha+\beta+\gamma-1} \end{aligned}$$

Therefore $U = \frac{X}{X+Y} \sim \text{Beta}(\alpha, \beta)$, $V = \frac{X+Y}{X+Y+Z} \sim \text{Beta}(\alpha + \beta, \gamma)$, $W = X + Y + Z \sim \text{Gamma}(\alpha + \beta + \gamma)$ and are independent.

Intuition behind the independence:

Say John waits X minutes for the Caltrain to DMV, then waits Z minutes at the line in DMV, and then waits Y minutes for the Caltrain from DMV to home. The above independence is saying that the fraction of time John spends on waiting for the first Caltrain (i.e. $\frac{X}{X+Y}$), is independent of the fraction of time he spends on waiting for the trains (i.e. $\frac{X+Y}{X+Y+Z}$), and is independent of his total waiting time (i.e. $X + Y + Z$).

Another view: Assume α, β, γ are integers. The sum $X + Y + Z$ is like the time until $k = \alpha + \beta + \gamma$ events have occurred in a Poisson process $N(t)$ (using the fact that the waiting times are i.i.d. exponential). A standard fact is that, conditional on the time t of the kth arrival, the previous $k - 1$ arrivals are distributed as the order statistics of $k - 1$ Uniform$(0, t)$ random variables, $U_{(1)}, \ldots, U_{(k-1)}$. In particular, for $k = \alpha + \beta + \gamma$, we have $X = U_{(\alpha)}, Y = U_{(\alpha+\beta)}$, and $Z = t$. Notice that the ratio $X/(X + Y)$ has a distribution which does not depend on t (to see this, multiply the top and bottom by $1/t$), so is independent of Z. The same can be seen for $(X + Y)/(X + Y + Z)$.

2

For any $y_1, \cdots, y_n$, let N_R, N_W, N_B be the total number of red, white and blue balls in these n balls, separately. we have

$$\begin{aligned} P(Y_1 = y_1, \cdots, Y_n = y_n) &\propto \int_{p_1+p_2+p_3=1} P(Y_1 = y_1, \cdots, Y_n = y_n|p) dp_1 dp_2 dp_3 \\ &= \int_{p_1+p_2+p_3=1} p_1^{N_R} p_2^{N_W} p_3^{N_B} dp_1 dp_2 dp_3 \\ &= \frac{\Gamma(N_k+1)\Gamma(N_W+1)\Gamma(N_B+1)}{\Gamma(n+3)} \end{aligned}$$

On the other hand,

$$\begin{aligned} & P(X_1 = y_1, \cdots, X_n = y_n) \\ =& P(X_1 = y_1)P(X_2 = y_2|X_1 = y_1) \cdots P(X_n = y_n|X_1 = y_1, \cdots, X_{n-1} = x_{n-1}) \\ =& \frac{N_R!N_W!N_B!}{3 \times 4 \times 5 \cdots (n+2)} \\ \propto& \frac{N_R!N_W!N_B!}{(n+2)!} \end{aligned}$$

The above two expression are equivalent since $\Gamma(k+1) = k!$ for $k \in \mathbb{N}^+$.

3

- To show S is finite almost surely, we only need to show $\mathbb{E}[S] < \infty$.

  Define the partial sum $S_N = \sum_{i=1}^{N} Z_i$. Since Z_i are non-negative, the sequence $\{S_N\}$ is monotonically non-decreasing, then by monotonic convergence theorem, we have

  $$\mathbb{E}[S] = \lim_{N\to\infty} \mathbb{E}[S_N] = \lim_{N\to\infty} \sum_{i=1}^{N} \mathbb{E}[Z_i] = A$$

  Thus $\mathbb{E}[S] < \infty$. It follows by the argument at the beginning that S is finite almost surely.

- To show $P = (P_0, P_1, \cdots)$ has a D_α distribution, we show that there exits a sequence of independent Gamma random variable $\{X_i\}$, with $X_i \sim \text{Gamma}(\alpha_i)$, such $P_i = \frac{X_i}{S_x}$, where $S_x = \sum_{i=1}^{\infty} X_i$. Moreover, $S_x \sim \text{Gamma}(\sum_{i=1}^{\infty} \alpha_1)$. (This can be proved by using moment generating function and convergence of moment generating function implies convergence in distribution, together with the fact that $\sum_{i=1}^{n} X_i \xrightarrow{d} S_x$. The proof is omitted as it is not the focus of this question)

We shall use the gamma representation of the Beta distribution, namely any Beta random variable B with parameter Beta(γ, δ) can be represented as $B = \frac{X}{X+Y}$, where $X \sim \text{Gamma}(\gamma)$, $Y \sim \text{Gamma}(\delta)$, X and Y are independent.

Now define $W_j = \frac{X_j}{X_j+V_j} = \frac{X_j}{\sum_{i \geq j} X_i}$, where $V_j = S_x - (X_1 + \cdots + X_j) \sim \text{Gamma}(A - \alpha_1 - \cdots - \alpha_j)$.

Using a similar argument as in problem 1 in this homework, we can show that

$$W_j \sim \text{Beta}(\alpha_j, \sum_{i \geq j+1} \alpha_i)$$

and $\{W_j\}$ is an independent sequence (Use change of variable one can show the joint density of $W_1, \cdots, W_n$ is the product of n independent factors for any n).

Now by the definition of the P_js, a simple inductive proof shows

$$P_0 = W_1 = \frac{X_1}{S_x}$$

$$P_1 = W_2(1 - P_0) = W_2(1 - W_1) = \frac{X_2}{S_x - X_1}(1 - \frac{X_1}{S_x}) = \frac{X_2}{S_x}$$

$$P_j = W_{j+1}(1 - P_0 - \cdots - P_{j-1}) = \frac{X_{j+1}}{S_x - X_1 - \cdots - X_j}(1 - \frac{X_1}{S_x} - \cdot - \frac{X_j}{S_x}) = \frac{X_{j+1}}{S_x}$$

which shows $P = (P_0, P_1, \cdots)$ has a D_α distribution by construction.