STATS 370: Problem Set #2

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Problem 1

We shall follow the notation of (Diaconis and Ylvisaker, 1979). Let $f_{n_0,x_0}(\theta) = e^{n_0 x_0 \theta - n_0 m(\theta)}$. Then, part (ii) of equation (2.6) of (Diaconis and Ylvisaker, 1979) states that

$$f''_{n_0,x_0}(\theta) = \left(n_0^2(m'(\theta) - x_0)^2 - n_0 m''(\theta)\right) f_{n_0,x_0}(\theta). \tag{1}$$

Integrating over Θ and using the fact that $\mathbb{E}[m'(\theta)] = x_0$, we find that

$$\int_\Theta f''_{n_0,x_0}(\theta) = z n_0^2 \mathrm{Var}[m'(\theta)] - z n_0 \mathbb{E}[m''(\theta)]. \tag{2}$$

Using a similar argument to the one given by (Diaconis and Ylvisaker, 1979) in the proof of Theorem 2, we can show that

$$\int_\Theta f''_{n_0,x_0}(\theta) = 0, \tag{3}$$

from which the desired conclusion follows.

Problem 2

First of all, note that $\theta|\bar{X} \sim \pi_{n_0+n,(n_0 x_0 + n\bar{x})/(n_0+n)}$, where π_{n_0,x_0} is the prior from Problem 1. The result of Problem 1 implies that

$$\mathrm{Var}[m'(\theta)|\bar{X}] = \frac{1}{n_0+n}\mathbb{E}[m''(\theta)|\bar{X}]. \tag{4}$$

Taking an expectation and using the law of iterated expectations, we get

$$\mathbb{E}[\mathrm{Var}[m'(\theta)|\bar{X}]] = \frac{1}{n_0+n}\mathbb{E}[\mathbb{E}[m''(\theta)|\bar{X}]] = \frac{1}{n_0+n}\mathbb{E}[m''(\theta)] = \frac{n_0}{n_0+n}\mathrm{Var}[m'(\theta)]. \tag{5}$$

The last equality follows again from Problem 1.

Problem 3

Since $\theta_1, \ldots, \theta_K$ are drawn independently from their respective priors, and the $\bar{X}_i$ are dependent only on θ_i, it follows that

$$\pi(\theta_1, \ldots, \theta_K | \bar{X}_1, \ldots, \bar{X}_K) = \prod_{i=1}^{K} \pi_i(\theta_i | \bar{X}_i); \tag{6}$$

i.e. the joint posterior of $(\theta_1, \ldots, \theta_K)$ decouples, and so $\theta_1, \ldots, \theta_K$ are independent a posteriori. Hence, we have

$$\begin{aligned} \mathbb{E}[\mathrm{Var}[\mu|\bar{X}_1, \ldots, \bar{X}_K]] &= \mathbb{E}\left[\mathrm{Var}\left[\sum_{i=1}^{K} \pi_i \mu_i | \bar{X}_1, \ldots, \bar{X}_K\right]\right] \\ &= \mathbb{E}\left[\sum_{i=1}^{K} \pi_i^2 \mathrm{Var}\left[\mu_i | \bar{X}_i\right]\right] \\ &= \sum_{i=1}^{K} \pi_i^2 \mathbb{E}\left[\mathrm{Var}\left[\mu_i | \bar{X}_i\right]\right] \\ &= \sum_{i=1}^{K} \pi_i^2 \mathrm{Var}[\mu_i] \frac{n_{0i}}{n_{0i} + n_i} \\ &= \sum_{i=1}^{K} \pi_i^2 \frac{b_i v_i}{n_i + b_i}, \end{aligned} \tag{7}$$

The penultimate equality follows from Problem 1, and the last equality follows from the fact (proved in Problem 2) that $n_{0i} = b_i = s_i^2 / v_i$.

Problem 4

The optimization problem we seek to solve is

$$\text{minimize} \quad \sum_{i=1}^{K} \pi_i^2 \frac{b_i v_i}{n_i + b_i}, \quad \text{subject to} \quad \sum_{i=1}^{K} c_i n_i \leq c, \tag{8}$$

where the minimization is over $(n_1, \ldots, n_K)$. Introducting a Lagrange multiplier λ, we find that the optimal vector will satisfy

$$\nabla_{n_1, \ldots, n_K} \left(\sum_{i=1}^{K} \pi_i^2 \frac{b_i v_i}{n_i + b_i} - \lambda \left(\sum_{i=1}^{K} c_i n_i - c \right) \right) = 0, \quad \sum_{i=1}^{K} c_i n_i = c. \tag{9}$$

The gradient equation reduces to

$$n_i = \frac{\pi_i \sqrt{b_i v_i}}{\sqrt{\lambda c_i}} - b_i, \quad i = 1, \ldots, K. \tag{10}$$

We can use these to solve for λ:

$$\frac{1}{\sqrt{\lambda}} = \frac{c + \sum_j b_j c_j}{\sum_j \sqrt{c_j \pi_j^2 s_j^2}}, \tag{11}$$

from which it follows that

$$n_i = \frac{\pi_i s_i (c + \sum_j b_j c_j)}{\sqrt{c_i} \sum_j \sqrt{c_j \pi_j^2 s_j^2}} - b_i, \quad i = 1, \ldots, K, \tag{12}$$

as desired.

Problem 5

Suppose that in stratum i, $X_{ij}|\theta_i \sim \mathcal{N}(\theta_i, \sigma_i^2)$, where each σ_i^2 is known. The conjugate prior is $\theta_i \sim \mathcal{N}(\mu_{0i}, \sigma_{0i}^2)$. Note that $v_i = \text{Var}[\mu_i] = \text{Var}[\theta_i] = \sigma_{0i}^2$. Next, we calculate that $s_i^2 = \mathbb{E}[\text{Var}[X_{ij}|\mu_i]] = \sigma_i^2$. Finally, $b_i = \sigma_i^2/\sigma_{i0}^2$. Since $n_{0i} = b_i$, sending $n_{i0} \to 0$ for all i is equivalent to $b_i \to 0$, and so the formula from Problem 4 becomes

$$n_i = \frac{\pi_i \sigma_i / \sqrt{c_i}}{\sum_j \pi_j \sigma_j \sqrt{c_j}} c, \tag{13}$$

which is the Neyman optimal allocation.