STATS 370 Problem Set 4 Solutions

Credit to Paulo Orenstein for providing these solutions.

1 Problem 1

We shall prove all four items via contradiction; that is, we assume the proposition is false and construct a contradiction by exhibiting a linear combination of the form $W = \sum_{i=1}^{n} c_i(X_i - P(X_i))$ such that $\sup W < 0$ or $\inf W > 0$. Note the inequalities below all hold almost surely.

(a) Say $X \leq 0$ and $P(X) > 0$. Then, taking $c > 0$, we see that $Y = c(X - P(X)) < 0$, no matter the outcome of X. This implies $\sup Y < 0$, and so the proposition holds.

(b) Assume $Y = cX$ and $P(Y) \neq cP(X)$. Without loss, take $P(Y) < cP(X)$, so we see that $Z = 1 \cdot (Y - P(Y)) - c(X - P(X)) = cP(X) - P(Y) > 0$ and thus $\inf Z > 0$. The case where $P(Y) > cP(X)$ gives $\sup Z < 0$. Hence, the proposition is true.

(c) Assume $X = c$ and $P(X) \neq$, say $P(X) > c$. Then $Y = (X - P(X)) < 0$ implies $\sup_Y < 0$. If we assume $P(X) < c$ instead, we get $Y = (X - P(X)) > 0$, so $\inf Y < 0$. Thus, we see the proposition holds.

(d) Take $X, Y, X + Y$ to be bounded quantities and assume $P(X + Y) \neq P(X) + P(Y)$, say $P(X + Y) < P(X) + P(Y)$. Then, $Z = (X + Y - P(X + Y)) - (X - P(X)) - (Y - P(Y)) = P(X) + P(Y) - P(X + Y) > 0$, so $\inf Z > 0$. It is easy to see that $P(X + Y) > P(X) + P(Y)$ leads to the case $\sup Z < 0$. This shows the proposition is true.

2 Problem 2

Assuming we are interested in minimizing the expected loss, since E is random, for the first loss we would like to solve $\min_\pi \mathbb{E}[(\mathbb{I}_E - \pi)^2]$. By first-order considerations, we arrive at the solution

$$\pi^* = \mathbb{E}[\mathbb{I}_E] = p,$$

so the subject is indeed impelled to declare his truthful evaluation of the probability of E, and therefore the rule is proper.

On the other hand, with the second loss we minimize $\min_\pi \mathbb{E}[|\mathbb{I}_E - \pi|]$, and it's easy to check the solution π^* is such that $P(X \leq \pi^*) \geq \frac{1}{2}$ and $P(X \geq \pi^*) \geq \frac{1}{2}$, that is, we get

$$\pi^* = \text{med}(\mathbb{I}_E) = \begin{cases} 1, & \text{if } P(E) > \frac{1}{2} \\ [0,1], & \text{if } P(E) = \frac{1}{2} \\ 0, & \text{if } P(E) < \frac{1}{2}. \end{cases}$$

Thus, we see from the solution above that the rule is not proper in general.

3 Problem 3

Here is the table with the respective values of p each one finds fair. Note there is a lot of variation due to their ideological positions, their sources of information, and how risk-averse they are.

Friend	p
A	.65
B	.7
C	.9
D	.6
E	.5
F	.4
G	.6
H	.85
I	.7
J	.4

As a casino, it would be better to play against those that seem very sure that Hillary will win – that is, those that are willing to pay high values of p. In this sense, Friend C might be a good person to play against as the casino.